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DIMENSIONAL REDUCTION BY A TWO-FORM
(another alternative to compactification)

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Abstract

It is shown that the local coupling of a higher dimensional graviton to a closed degenerate two-form produces dimensional reduction by spontaneous breakdown of extra-dimensional translational symmetry. Four dimensional Poincaré invariance emerges as residual symmetry. As a specific example, a six dimensional geometry coupled to a closed rank 2 two-form yields the 'ground state'

$$ds^2 = \mathrm{e}^{-|\xi|^2/4l^2}\eta_{\mu\nu}dx^\mu dx^\nu + \delta_{ij}d\xi^i d\xi^j$$

with l a fundamental length scale. At low energies, space-time reduces to four observable dimensions and general relativistic gravity is reproduced.

There has been recently a renewed interest in higher dimensional geometry as an intermediate step in between today's high energies field theories and a foretold final theory. Besides the search for phenomenological implications, one of the hottest problems remains that of reproducing the effective four dimensional nature of space-time at low energies scales. To one side the traditional hypothesis of compact extra dimensions has been fully reconsidered [1]. To the other, a remarkable alternative has been put forward by showing that dimensional reduction take place without compactification in a set-up with a 3-brane embedded in the higher dimensional space [2]. In both approaches dimensional reduction is produced by global topological assumptions –compactness of extra-dimensions in the first case, 3-brane boundary conditions in the second one.

In this letter we explore an alternative based on the local interaction with a closed degenerate two-form. The specific model we consider is a six dimensional space-time (M^6, g_{IJ}) on which a closed rank 2 two-form B_{IJ} is defined[1]. No physical hypothesis are made on the nature of B_{IJ}. In solving generalized Einstein's equations it is found that translational invariance is spontaneously broken and four dimensional Poincaré invariance emerges as the residual symmetry of the geometric 'ground state'. The corresponding metric tensor reproduces –up to the functional form of the wrap factor and two instead of one extra dimensions– the one recently investigated by L. Randall and R. Sundrum [2]. On low energies scales space-time dynamically reduces to four effective dimensions and the ordinary general relativistic picture is recovered.

[1]Capital Latin indices run over $I = 0, 1, ..., 5$

Field Equations. The reason for such a behavior grounds in the peculiar properties of the local coupling with a closed degenerate two-form. In looking for field equations we first observe that skew-symmetry and closure condition $(d\mathrm{B})_{IJK} = 0$ make B_{IJ} very similar to an ordinary electromagnetic field. In particular, it is possible to introduce a vector potential A_I which six dimensional curl equals B_{IJ}

$$(d\mathrm{A})_{IJ} = \mathrm{B}_{IJ} \tag{1}$$

As in ordinary electromagnetism A_I is determined up to the gradient of an arbitrary function only. The transformation $\mathrm{A}_I \to \mathrm{A}_I + \partial_I \chi$ leaves B_{IJ} unchanged. By this analogy, we are lead to include in the Lagrangian density of the theory a term proportional to $\mathrm{B}_{IJ}\mathrm{B}^{IJ}$ –besides the scalar curvature R and the cosmological term. Other contributions containing B_{IJ} or its covariant derivatives may be assumed higher order. At this point however, we should recall that the four dimensional electromagnetic analogy is purely formal. In a higher dimensional space-time the one-form A_I still can be coupled to geometry by giving a 'charge' to the graviton. By extending a well established strategy, g_{IJ} is promoted to a complex field[2] and everywhere in field equations ordinary derivatives ∂_I are replaced by gauge covariant ones

$$\partial_I - i l^{-2} \mathrm{A}_I \tag{2}$$

where (by choosing B_{IJ} dimensionless) l carries the dimension of a length. Clearly, by a redefinition of vector potential $\mathrm{A}_I \to \mathrm{A}_I + \partial_I \chi$ the metric field is assumed to transform according to $\mathrm{g}_{IJ} \to \mathrm{e}^{i\chi/l^2}\mathrm{g}_{IJ}$.
In this way we come to the following generalization of Einstein's field equations

$$\not{\mathrm{R}}_{IJ} - \frac{1}{2}\not{\mathrm{R}}\,\mathrm{g}_{IJ} = \Lambda\,\mathrm{g}_{IJ} - K\left(\mathrm{B}_I{}^K\mathrm{B}_{JK} - \frac{1}{4}\mathrm{B}_{KL}\mathrm{B}^{KL}\mathrm{g}_{IJ}\right) \tag{3}$$

where Λ and K are constants, g_{IJ} is now a complex field and Dirac's slash notation has been borrowed to denote quantities in which ordinary derivatives are replaced by gauge covariant ones.

Adapted Coordinates. A remarkable property of Eqns.(3) is that for no choice of Λ, K and no A_I gauge, the flat space-time Minkowskian metric η_{IJ} is a solution. To prove this and solve field equations it is very convenient to work in an adapted coordinates system. Given the closure condition $(d\mathrm{B})_{IJK} = 0$, a classical theorem of Darboux ensures the possibility of globally finding coordinates in such a way that

$$B_{IJ} = \begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & & & & \\ & & 0 & 0 & & \\ & & 0 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} \tag{4}$$

Further assuming B_{IJ} non null directions to be space-like, Darboux coordinates are denoted by $\mathrm{x}^I = (x^\mu, \xi^i)$ with $\mu = 0, 1, 2, 3$ and $i = 4, 5$. In such coordinates frames A_I

[2]As for an ordinary charged scalar or fermion field, the complex nature of the metric tensor has nothing to do with its space-time transformation properties.

can be chosen in the form $\mathrm{A}_I = (0, A_i)$ with A_i the vector potential of a two-dimensional homogeneous magnetic field –e.g. in the symmetric gauge $A_i = (\xi^5/2, -\xi^4/2)$.

Four Effective Dimensions. Assuming the geometric 'ground state' to be compatible with four dimensional Poincaré invariance, we are lead to the following ansatz for the six dimensional line element [2]

$$ds^2 = \phi(\xi)\eta_{\mu\nu}dx^\mu dx^\nu + \delta_{ij}d\xi^i d\xi^j \tag{5}$$

With this choice field equations take the form

$$\frac{1}{l^2}\left(\partial\!\!\!/_4\partial\!\!\!/_4 + \partial\!\!\!/_5\partial\!\!\!/_5\right)\phi\,\eta_{\mu\nu} = \frac{1}{3}\left(2\Lambda - K\right)\phi\,\eta_{\mu\nu} \tag{6}$$

$$\frac{1}{l^2}\,\partial\!\!\!/_5\partial\!\!\!/_5\phi - \frac{i}{2l^3}\,A_4\,\partial\!\!\!/_4\phi + \frac{i}{2l^3}\,A_5\,\partial\!\!\!/_5\phi + \frac{3}{4l^2}\frac{(\partial\!\!\!/_4\phi)^2}{\phi} + \frac{1}{4l^2}\frac{(\partial\!\!\!/_5\phi)^2}{\phi} = \frac{K}{4}\,\phi \tag{7}$$

$$\frac{1}{l^2}\left(\partial\!\!\!/_4\partial\!\!\!/_5 + \partial\!\!\!/_5\partial\!\!\!/_4\right)\phi + \frac{i}{l^3}\,A_4\,\partial\!\!\!/_5\phi + \frac{i}{l^3}\,A_5\,\partial\!\!\!/_4\phi - \frac{1}{l^2}\frac{(\partial\!\!\!/_4\phi)\,(\partial\!\!\!/_5\phi)}{\phi} = 0 \tag{8}$$

$$\frac{1}{l^2}\,\partial\!\!\!/_4\partial\!\!\!/_4\phi + \frac{i}{2l^3}\,A_4\,\partial\!\!\!/_4\phi - \frac{i}{2l^3}\,A_5\,\partial\!\!\!/_5\phi + \frac{1}{4l^2}\frac{(\partial\!\!\!/_4\phi)^2}{\phi} + \frac{3}{4l^2}\frac{(\partial\!\!\!/_5\phi)^2}{\phi} = \frac{K}{4}\,\phi \tag{9}$$

where dimensionless operators $\partial\!\!\!/_i = l\partial_i - il^{-1}A_i$ have been introduced. $\partial\!\!\!/_4$ and $\partial\!\!\!/_5$ have the form of kinematical momenta for a charged particle coupled to a homogeneous magnetic field. They close canonical commutation relations

$$[\partial\!\!\!/_4, \partial\!\!\!/_5] = i \tag{10}$$

By this analogy eq.(6) is identified with the wave equation of a scalar particle in a two dimensional homogeneous magnetic field. All solutions are easily constructed. No constant is included. By replacing them in the remaining equations we find –up to a normalization factor $\mathcal{N}$ and an arbitrary point $\bar{\xi}$ in extra directions– the only solution

$$\phi(\xi) = \mathcal{N}\mathrm{e}^{-|\xi-\bar{\xi}|^2/4l^2} \tag{11}$$

with $|\xi|^2 = {\xi^4}^2 + {\xi^5}^2$ and corresponding to the values $\Lambda = -5/2l^2$ and $K = -2/l^2$. Conventionally, we set $\mathcal{N} = 1$ and choose $\bar{\xi}$ as the origin of extra coordinates. In this way the line element relative to the geometric 'ground state' is univocally determined as

$$ds^2 = \mathrm{e}^{-|\xi|^2/4l^2}\eta_{\mu\nu}dx^\mu dx^\nu + \delta_{ij}d\xi^i d\xi^j \tag{12}$$

Translational symmetry in extra directions is spontaneously broken. At energy scales small compared to the one associated to the length l, space-time is effectively squeezed on four ordinary directions while the two extra ones are made inaccessible by the Gaussian factor $\mathrm{e}^{-|\xi|^2/4l^2}$. Up to the functional form of the wrap factor $\phi(\xi)$ and two instead of one extra dimensions, the metric we find resembles the one recently studied by Randall and Sundrum. The main difference in the two approaches is that in ref.[2] translational invariance in extra directions is broken from the very beginning by assuming the existence of the 3-brane while here it is spontaneously broken by assuming the local interaction with the closed degenerate two-form B_{IJ}. Apart form this, the low energy scenario associated to the metric tensor (12) shares most of the characteristics emphasized by Randall and Sundrum. We refer to their papers for a detailed discussion.

Four Dimensional Gravity. By a slight modification we can easily construct solutions corresponding to a gravitational field in the effective four dimensional space-time. In (5), we replace the flat four dimensional metric $\eta_{\mu\nu}$ by an arbitrary metric $g_{\mu\nu}(x)$

$$ds^2 = \phi(\xi) g_{\mu\nu}(x) dx^\mu dx^\nu + \delta_{ij} d\xi^i d\xi^j \tag{13}$$

By replacing this new ansatz in field equations we are lead to the differential problem

$$\frac{1}{l^2}\left(\partial\!\!\!/_4\partial\!\!\!/_4 + \partial\!\!\!/_5\partial\!\!\!/_5\right)\phi\, g_{\mu\nu} + \frac{2}{3}\left(R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}\right) = \frac{1}{3}\left(2\Lambda - K\right)\phi\, g_{\mu\nu} \tag{14}$$

$$\frac{1}{l^2}\,\partial\!\!\!/_5\partial\!\!\!/_5\phi - \frac{i}{2l^3}\,A_4\,\partial\!\!\!/_4\phi + \frac{i}{2l^3}\,A_5\,\partial\!\!\!/_5\phi + \frac{3}{4l^2}\frac{(\partial\!\!\!/_4\phi)^2}{\phi} + \frac{1}{4l^2}\frac{(\partial\!\!\!/_5\phi)^2}{\phi} - \frac{1}{4}R = \frac{K}{4}\,\phi \tag{15}$$

$$\frac{1}{l^2}\left(\partial\!\!\!/_4\partial\!\!\!/_5 + \partial\!\!\!/_5\partial\!\!\!/_4\right)\phi + \frac{i}{l}\,A_4\,\partial\!\!\!/_5\phi + \frac{i}{l}\,A_5\,\partial\!\!\!/_4\phi - \frac{1}{l^2}\frac{(\partial\!\!\!/_4\phi)\,(\partial\!\!\!/_5\phi)}{\phi} = 0 \tag{16}$$

$$\frac{1}{l^2}\,\partial\!\!\!/_4\partial\!\!\!/_4\phi + \frac{i}{2l^3}\,A_4\,\partial\!\!\!/_4\phi - \frac{i}{2l^3}\,A_5\,\partial\!\!\!/_5\phi + \frac{1}{4l^2}\frac{(\partial\!\!\!/_4\phi)^2}{\phi} + \frac{3}{4l^2}\frac{(\partial\!\!\!/_5\phi)^2}{\phi} - \frac{1}{4}R = \frac{K}{4}\,\phi \tag{17}$$

where $R_{\mu\nu}$ and R denote respectively Riemann tensor and scalar curvature associated to $g_{\mu\nu}$. The problem exactly separates in the one we just solved for the freedom $\phi(\xi)$ plus a standard Einstein problem for the four dimensional metric $g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = 0 \tag{18}$$

While on energy scales of order l^{-2} space-time is squeezed on a 1+3 pseudo-Riemannian hyper-surface, the low energy –order one– dynamics of the effective four dimensional geometry is described by Einstein field equations.

Other Fields. In concluding, we remark that the coupling of B_{IJ} to other higher dimensional 'charged' fields –either bosonic or fermionic– also produces their squeezing on the effective four dimensional space-time by a very similar mechanism [3].

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References

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